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EUCLIDEAN FUNCTIONAL INTEGRAL APPROACH FOR DISORDER VARIABLES AND KINKS

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We propose and investigate an euclidean functional integral approach for the construction of local kink operators and disorder variables. The main difference from the existing quasiclassical approach is the emphasis on local fields instead of kink states and collective coordinate methods. We show that all two-dimensional kink fields emerge from the gauge theory of matter fields in Bohm-Aharonov fluxes. In the case of real scalar or self-conjugate Majorana fields the language of "flat non-trivial fibre bundles" is unavoidable since there is no minimal coupling to a vector potential. As an application we reproduce the construction of the Ising field theory and related models. In this way we obtain a unified treatment of the work of Sato et al. with that of other authors.

1. Introduction

Kinks and solitons entered QFT at the beginning of the 70's [1]. Solutions of classical non-linear field equations were incorporated as particle states into QFT via quasiclassical methods [2]. Unless the classical field theory possesses special properties which stabilize those objects against quantum fluctuations, there will be no reason to believe that such classical objects have a counterpart in QFT. One such

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special feature is the existence of an infinite number of higher conservation laws which is thought to be related to the integrability of the system. The exactness of the quasiclassical spectrum of the H-atom in QM or that of the sine-Gordon equation in QFT are well-known illustrations. Another, and perhaps more general mechanism, leading to this stability, is topology entering through homotopy classes. In this work we will attempt to lay the foundation for a functional integral approach for these topological objects which we refer to as topological solitons or kinks and which are related to disorder variables of statistical mechanics. The quasiclassical approach to those objects aims mainly at the construction of particle states carrying new quantum numbers and hence lying in different (superselecting) sectors than the vacuum. The mathematical method is that of introducing collective coordinates and it originated from nuclear physics [3]. This method is somewhat at odds with the spirit of QFT. It could be more appropriate to construct (local) interpolating fields and to delegate the particle discussion to the LSZ asymptotic behaviour of these fields. In this work we will show that the construction of such kink fields for two-dimensional QFT in the euclidean functional approach will lead to the statistical mechanics of matter fields (scalar and spinor fields) within Bohm-Aharonov [4] fluxes. The relativistic invariance of the correlation functions of kinks originates simply from the gauge invariance of the Bohm-Aharonov functional determinants. Our construction is related to the order-disorder duality of Kadanoff [5] and the resulting (in the physical points) dual algebra of 't Hooft [6], the space-like commutation relations of the kink operators with the original fields being a consequence of this duality. Whether new "particle sectors" exist or not is related to the existence of a broken symmetry phase of the model. The application of these general ideas to very special two-dimensional models already gives a wealth of results. Among other things we will show that the disorder and order variables of the Lenz-Ising field theory emerge from the study of ordinary and γ^5 (axial) Bohm-Aharonov potentials for free massive spinor fields. This model also furnishes a natural illustration for a situation envisaged by Wu and Yang [7]: the language of fibre-bundle theory becomes unavoidable in its euclidean functional integral construction. Our approach will reproduce and unify the results of Sato, Miwa and Jimbo [17] with those of Lehman and Stehr [9] and Schroer, Truong and Weisz [10]. It also exposes the deeper reason behind the "doubling" of the Ising model which yields a simple formalism [11] for the derivation of its short-distance properties (e.g. critical exponents). A perturbative systematic for kinks (disorder variables) in the broken symmetry phase of the ϕ^4 theory and Z_N lagrangian as well as generalizations to higher dimensions will be left for future publications.

The material is organized as follows: In sect. 2, we derive expressions for disorder parameters in the case of scalar and spinor fields. In sects. 3 and 4, we compute these parameters, in the context of free fields. In sect. 5 we construct explicitly the kink operators for scalar and spinor fields. In sect. 6 we study the case of real scalar or Majorana spinor fields. In sect. 7 we generalize to non-abelian kinks. Concluding remarks are presented in sect. 8.

2. Determination of local covariant disorder variables

2.1. SCALAR FIELDS

Let us consider the theory of a complex scalar field, defined by the lagrangian density

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - M^2\phi^*\phi + U(\phi, \phi^*), \tag{2.1}$$

where U is a polynomial.

Choosing $\phi(x)$ as our order parameter, we want to obtain the disorder field, $\mu(x)$, that obeys the usual, Z_N, dual algebra commutation relations

$$\mu(x,t)\phi(y,t) = e^{i(2\pi/N)\theta(y-x)}\phi(y,t)\mu(x,t), \tag{2.2a}$$

$$\mu(x,t)\pi(y,t) = e^{i(2\pi/N)\theta(y-x)}\pi(y,t)\mu(x,t), \tag{2.2b}$$

where $\theta(x)$ is the step function.

There are two ways of doing this.

The first one is just to guess formally what operator has the commutation rule (2.2). We can easily see that such an operator is (assuming formally canonical commutation relations)

$$\mu(x) = \exp\left[\frac{2\pi}{N}\int_{x,C}^{\infty}\varepsilon^{\mu\nu}(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi)\,dz_\mu\right]. \tag{2.3}$$

This method has the disadvantage that the obtained field is not path independent and, when computing correlation functions involving μ, one should introduce string renormalization counterterms, in order to get path independence.

The second and more rigorous way is the field theoretical [12] generalization of the method of Kadanoff and Ceva [5], for computation of disorder variables' correlation functions in the Ising model. This procedure [12] gives us an already path-independent result. Let us apply it for the present theory.

The first step is to assume that the correlation function $\langle\mu(x)\mu^*(y)\rangle$ is obtained by "deforming" the action along the curve C, joining x and y, in the euclidean functional integral,

$$\langle\mu(x)\mu^*(y)\rangle_C = N\int[D\phi][D\phi^*]\exp\left\{-\int d^2z\left[\partial_\mu\phi^*\partial_\mu\phi + M^2\phi^*\phi + U(\phi,\phi^*)\right.\right.$$
$$\left.\left. + a\partial_\nu\psi\int_{x,C}^{y}\varepsilon^{\mu\nu}\delta(z-\xi)\,d\xi_\mu + \mathcal{L}_C(\phi,\phi^*)\right]\right\}, \tag{2.4}$$

where $\psi(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*)$ is a generic functional to be determined, $\mathcal{L}_C(\phi, \phi^*)$ is a path renormalization counterterm and N is the usual normalization constant.

The second step is to take path independence as a principle and use the symmetry we are interested in, in order to determine explicitly $\partial_\nu\psi$ and $\mathcal{L}_C$.

In the case of Thirring model, where we have the continuous symmetry $\phi \to \phi + a$, $\psi = \phi$ and $\mathcal{L}_C$ is independent of ϕ [12].

Writing

$$\int_{x,C}^{y} \varepsilon^{\mu\nu}\delta(z-\xi)\,d\xi_\mu = \int_{x,C'}^{y} \varepsilon^{\mu\nu}\delta(z-\xi)\,d\xi_\mu + \oint_\Gamma \varepsilon^{\mu\nu}\delta(z-\xi)\,d\xi_\mu, \tag{2.5}$$

where $\Gamma = C - C'$, and making the change of variable $\phi \to e^{i2\pi/N}\phi$ inside the region S bounded by Γ, we obtain

$$\begin{aligned}\langle\mu(x)\mu^*(y)\rangle_C = N\int[D\phi][D\phi^*]\exp\Bigg\{-\int d^2z\Bigg[&\partial_\mu\phi^*\partial_\mu\phi + M^2\phi^*\phi + U(\phi,\phi^*)\\
&+\mathcal{L}_C(\phi,\phi^*) + a\partial_\nu\psi\int_{x,C'}^{y}\varepsilon^{\mu\nu}\delta(z-\xi)\,d\xi_\mu\\
&+a\partial_\nu\psi\oint_\Gamma\varepsilon^{\mu\nu}\delta(z-\xi)\,d\xi_\mu + a\delta(\partial_\nu\psi)\int_{x,C}^{y}\varepsilon^{\mu\nu}\delta(z-\xi)\,d\xi_\mu\\
&+\left(\frac{2\pi}{N}\right)^2\phi^*\phi\oint_\Gamma\oint_\Gamma\delta(z-\xi)\delta(z-\eta)\,d\xi_\mu d\eta_\mu\\
&+i\frac{2\pi}{N}(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi)\oint_\Gamma\varepsilon^{\mu\nu}\delta(z-\xi)\,d\xi_\mu\Bigg]\Bigg\},\end{aligned} \tag{2.6}$$

where $\delta(\partial_\nu\psi)$ represents the variation of $\partial_\nu\psi$ with respect to the symmetry operation. $\mathcal{L}_C$ is assumed to be invariant under this operation.

Eliminating terms proportional to one closed integral, we get

$$a\partial_\nu\psi = -i\frac{2\pi}{N}(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi), \tag{2.7}$$

and using the fact that $\phi \to e^{i(2\pi/N)\theta(s)}\phi$, we find

$$a\delta(\partial_\nu\psi) = -2\left(\frac{2\pi}{N}\right)^2\phi^*\phi\oint_\Gamma\varepsilon^{\alpha\nu}\delta(z-\xi)\,d\xi_\alpha. \tag{2.8}$$

Inserting this result in (2.6) and using the delta-function properties, we arrive at the conclusion that

$$\mathcal{L}_C(\phi,\phi^*) = \left(\frac{2\pi}{N}\right)^2 \phi^*\phi \int_{x,C}^{y}\int_{x,C}^{y} \delta(z-\xi)\delta(z-\eta)\,\mathrm{d}\xi_\mu \mathrm{d}\eta_\mu \tag{2.9}$$

ensures path independence.

The form of the path independent, two-point disorder correlation function is, therefore,

$$\begin{aligned}\langle\mu(x)\mu^*(y)\rangle = N\int[\mathrm{D}\phi][\mathrm{D}\phi^*]\exp\Bigg\{-\int \mathrm{d}^2z\Bigg[&\mathcal{L} - i\frac{2\pi}{N}(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi)\\ &\times\int_{x,C}^{y}\varepsilon^{\mu\nu}\delta(z-\xi)\,\mathrm{d}\xi_\mu + \left(\frac{2\pi}{N}\right)^2\phi^*\phi\\ &\times\int_{x,C}^{y}\int_{x,C}^{y}\delta(z-\xi)\delta(z-\eta)\,\mathrm{d}\xi_\mu \mathrm{d}\eta_\mu\Bigg]\Bigg\},\end{aligned} \tag{2.10}$$

from which we can draw the formal euclidean disorder variable

$$\mu(x) = \exp\left[i\frac{2\pi}{N}\int_{x,C}^{\infty}\varepsilon^{\mu\nu}(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi)\,\mathrm{d}z_\mu\right], \tag{2.11}$$

which corresponds to the Minkowski one, (2.3).

Without the last bilinear part in (2.10) this disorder variable would not be path independent, i.e. (2.11) would not define a scalar euclidean field.

Let us now introduce an external vector field A_μ, given by

$$A_\mu = 2\pi\int_{x,C}^{y}\varepsilon^{\mu\nu}\delta(z-\xi)\,\mathrm{d}\xi_\nu. \tag{2.12}$$

We can, then, write (2.10) as

$$\langle\mu(x)\mu^*(y)\rangle = N\int[\mathrm{D}\phi][\mathrm{D}\phi^*]\exp\left[-\int\mathrm{d}^2z\left\{(D_\mu\phi)^*(D_\mu\phi) + M^2\phi^*\phi + U(\phi,\phi^*)\right\}\right], \tag{2.13}$$

where $D_\mu = \partial_\mu - i\alpha A_\mu$, $\alpha = 1/N$.

The two-point disorder correlation function, can be described by a minimal coupling of ϕ with the external field A_μ. In this formulation, path independence is a natural consequence of gauge invariance, since a path deformation is equivalent to a gauge transformation:

$$A_{\mu,C'} - A_{\mu,C} = 2\pi \oint_\Gamma \varepsilon^{\alpha\mu}\delta(z-\xi)\,d\xi_\alpha = 2\pi\partial_\mu\theta(S). \tag{2.14}$$

In the same way, we can introduce order-disorder correlation functions

$$\langle \mu(x_1)\mu^*(y_1)\phi(x_2)\phi^*(y_2)\rangle$$
$$= N\int [D\phi][D\phi^*]\exp\left[-\int d^2z\left\{\mathcal{L}\left[\phi,\phi^*,D_\mu\phi,(D_\mu\phi)^*\right]\right\}\right]\phi(x_2)\phi^*(y_2). \tag{2.15}$$

Now global path invariance is lost. The correlation function is multiplied by the factor $e^{i2\pi/N}$ ($e^{-i2\pi/N}$), every time the path is deformed over $\phi(x_2)$ ($\phi^*(y_2)$).

Our result, at first sight surprising, that the properties of disorder variables are described by a gauge theory of matter fields interacting with Bohm-Aharonov potentials is in full accordance with the ideas of 't Hooft and Kadanoff [5, 6] about disorder variables. One introduces disorder variable correlation functions by coupling the matter fields with external gauge potentials ("deformation" of the action). The phase ambiguity found in euclidean order-disorder correlation functions reflects itself in the dual algebra commutation relations existing in the Minkowski region, as was well established in the case of Thirring and Schwinger models [12].

2.2. SPINOR FIELDS

Let us now treat the theory of a two-dimensional spinor field defined by the lagrangian density

$$\mathcal{L} = -i\bar\psi\partial\!\!\!/\psi + M\bar\psi\psi + U(\psi,\bar\psi), \tag{2.16}$$

where U is a general polynomial in ψ and $\bar\psi$.

We want to introduce the disorder parameter appropriate for this theory, using the same method applied previously to scalar fields.

We write, therefore, in the euclidean region,

$$\langle\mu(x)\rangle = N\int[D\psi][D\bar\psi]\exp\left[-\int d^2z\left\{i\bar\psi D\!\!\!/\psi - iM\bar\psi\psi + U(\psi,\bar\psi)\right\}\right], \tag{2.17}$$

where $D_\mu = \partial_\mu - i\alpha A_\mu$, $\alpha = 1/N$ and

$$A_\mu = 2\pi\int_{x,C}^{\infty}\varepsilon^{\mu\nu}\delta(z-\xi)\,d\xi_\nu. \tag{2.18}$$

Again, locality of $\langle\mu\rangle$ is a consequence of the gauge invariance of the theory. Observe that in the spinor case, due to the simpler structure of the current, we do not have the analog of the last bilinear part of (2.10).

A problem now arrives, of how to determine the order parameter $\langle\sigma\rangle$, dual to $\langle\mu\rangle$, in the present case. The solution is simple: in order to define $\langle\sigma\rangle$, we couple the matter field with $\tilde{A}_\mu=\gamma^5 A_\mu$, the dual of the field A_μ, eq. (2.18). We are, thus, led to an axial gauge theory.

Since in the euclidean spinor case, both ψ and $\bar{\psi}$ transform in the same way under a chiral U(1) or (Z_N) transformation, when showing path independence of $\langle\sigma\rangle$, along the same lines as the scalar case (2.5)–(2.10), the mass term restricts α to the only allowed value $\alpha=\frac{1}{2}$. The same phenomenon already occurred in the sine-Gordon version of the massive Thirring model [12].

3. The free massive complex scalar field

The truly interesting two-dimensional models are those which possess an ordered phase, for example the ϕ^4 interaction in the Higgs phase. For such models one would like to show that the vacuum expectation value $\langle\mu\rangle$ vanishes. The physical reason for this should be that the energy of a matter field submitted to an Aharonov-Bohm flux in a non-trivial vacuum diverges logarithmically with the radius of the volume.

Furthermore, one would like to study the 3-point function $\langle\mu(x)\mu^*(y)\phi(z)\rangle$ in a lowest order perturbation systematics, in order to make contact with the known semiclassical approach, which, instead of dealing with interpolating fields μ, approximates the form factor of ϕ in the particle-kink states.

We leave these discussions for future publication and calculate μ in the free field phase in which it condenses ($\langle\mu\rangle\neq 0$) and therefore does not interpolate kink particles.

We will consider the case of a free massive complex scalar field. In this case, we have

$$\langle\mu(x)\rangle=N\int[\mathrm{D}\phi][\mathrm{D}\phi^*]\exp\left[-\int \mathrm{d}^2z\left\{(D_\mu\phi)^*(D_\mu\phi)+M^2\phi^*\phi\right\}\right] \tag{3.1}$$

with A_μ given by (2.18). N^{-1} is the same functional integral with ∂_μ instead of D_μ.

We see, therefore, after a trivial functional integration, that

$$\langle\mu(x)\rangle=\mathrm{Det}\left[\frac{-\partial^2+M^2}{-D^2+M^2}\right]. \tag{3.2}$$

In order to compute the determinant, we introduce a circular boundary of radius R and impose Dirichlet boundary conditions over it.

It is more convenient to work in the "vortex gauge", defined by

$$\phi(r,\varphi)\to e^{-i\alpha\varphi}\phi(r,\varphi), \qquad \alpha=1/N. \tag{3.3}$$

In this gauge, the Bohm-Aharonov operator, eigenvalue equation is given by

$$\left[-\left(\partial_r^2+\frac{1}{r}\partial_r\right)-\frac{1}{r^2}(\partial_\varphi+i\alpha)^2+M^2\right]\phi=\lambda^2\phi. \tag{3.4}$$

Requiring integrability at the origin, we find the eigenfunctions

$$\phi_{m,n}=\frac{1}{\sqrt{2\pi}}e^{im\varphi}J_{|m+\alpha|}(k_{mn}r), \qquad m=0,\pm1,\dots, \qquad n=1,2,\dots, \tag{3.5}$$

where $k_{mn}R$ are the zeros of the Bessel function $J_{|m+\alpha|}$.

The Bohm-Aharonov eigenvalues are $\lambda^2_{m,n}=k^2_{m,n}+M^2$.

We can, then, write (3.2), in the form

$$\langle\mu\rangle=\exp-\left\{\sum_{n=1}^{\infty}\ln\frac{k_{0n}^{+2}+M^2}{k_{0n}^2+M^2}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\ln\frac{\left(k_{mn}^{+2}+M^2\right)\left(k_{mn}^{-2}+M^2\right)}{\left(k_{mn}^2+M^2\right)\left(k_{mn}^2+M^2\right)}\right\}, \tag{3.6}$$

where $k^+_{m,n}$, are related to the zeros of $J_{|m+\alpha|}$ for $m\geqslant0$, $k^-_{m,n}$ to the zeros of $J_{|m+\alpha|}$, for $m<0$ and $k_{m,n}$ are related to the zeros of the "free" Bessel functions $J_{|m|}$.

Observing that large values of m should not contribute to (3.6), we can use the asymptotic form of the zeros of a Bessel function J_ν [13],

$$x_{\nu,n}=\left(n+\tfrac12\nu-\tfrac14\right)\pi-\frac{4\nu^2-1}{8\left(n+\frac12\nu-\frac14\right)\pi}+\cdots, \tag{3.7}$$

in order to obtain

$$k^{\pm}(k)=k\pm\frac{\pi\alpha}{2R}-\frac{\alpha^2\pm2\alpha m}{2kR^2}+\cdots. \tag{3.8}$$

Inserting this expression in (3.6) and using the well-known relation for $R\to\infty$,

$$\frac{1}{\pi R^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\to\frac{1}{(2\pi)^2}\int_0^{\pi}d\varphi\int_0^{\infty}k\,dk,$$

$$\frac{1}{R}\sum_{n=1}^{\infty}\to\frac{1}{\pi}\int_0^{\infty}dk, \qquad (m=0), \tag{3.9}$$

we get, after integration,

$$\langle\mu\rangle=\left(\frac{M^2}{\Lambda^2}\right)^{[\alpha/2+([2-\pi^2]/16)\alpha^2]}, \tag{3.10}$$

where Λ is a momentum cutoff. The α term comes from the first term in (3.6) and the α^2 term from the second.

For the range $0<\alpha<1$, this expression approaches zero which is consistent with the expectation that outside perturbation theory the wave function renormalization factor, Z approaches zero.

The power behaviour of Z, on the other hand, indicates that the disorder variable will be a renormalizable local field. Considering the next terms in (3.7) and (3.8) would not change the power dependence on Λ, but only introduce smaller corrections in the numeric coefficient in (3.10).

Choosing Z appropriately, one can set $\langle\mu\rangle=1$.

The order parameter $\langle\sigma\rangle=\langle\phi\rangle$, is trivially seen to be zero.

4. The free massive spinor field

Again, as we did in the boson case, we are going to compute in this section the order and disorder parameters, $\langle\sigma\rangle$ and $\langle\mu\rangle$, for the free theory of massive spinor fields.

Let us start with the disorder variable, which in this case is given by

$$\langle\mu\rangle=N\int[\mathrm{D}\psi][\mathrm{D}\bar{\psi}]\exp\left[-\int \mathrm{d}^2z\{i\bar{\psi}\not{D}\psi-iM\bar{\psi}\psi\}\right], \tag{4.1}$$

where A_μ is expressed by (2.18).

Doing the integration, we get

$$\langle\mu\rangle=\mathrm{Det}\left[\frac{i\not{D}-iM}{i\not{\partial}-iM}\right]. \tag{4.2}$$

Again, in order to compute the above determinant, we introduce a circular boundary of radius R, and are concerned with the eigenvalue problem

$$(i\not{D}-iM)\psi=\lambda\psi, \qquad \lambda=\lambda_0-iM. \tag{4.3}$$

It has been noted in a different context [14] that Dirichlet boundary conditions for this problem destroy charge-conjugation invariance of the theory. The correct boundary conditions are shown to be the "spectral boundary conditions" [14], which happen to be also the appropriate ones for the Atiyah-Singer index theorem with a boundary term.

We can write (4.3) as

$$\begin{pmatrix} -iM & -D_1+iD_2 \\ D_1+iD_2 & -iM \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{4.4}$$

or

$$-D^2\psi_1 = \lambda_0^2\psi_1, \tag{4.5a}$$

$$\psi_2 = (1/\lambda_0)(D_1+iD_2)\psi_1. \tag{4.5b}$$

For each ψ, there exists a $\psi' = \gamma^5\psi$, such that

$$(-i\not{D} - iM)\psi' = \lambda'\psi', \qquad \lambda' = -\lambda_0 - iM, \tag{4.6}$$

and whose components satisfy the same equations (4.5).

In the "vortex gauge", defined by (3.3), the integrable solutions of (4.5) read

$$\psi_m = \frac{1}{\sqrt{4\pi}}\begin{pmatrix} e^{im\varphi}J_{m+\alpha}(kr) \\ -e^{i(m+1)\varphi}J_{m+\alpha+1}(kr) \end{pmatrix}, \qquad m \geqslant 0,$$

$$\psi_m = \frac{1}{\sqrt{4\pi}}\begin{pmatrix} e^{im\varphi}J_{-m-\alpha}(kr) \\ -e^{i(m+1)\varphi}J_{-m-\alpha-1}(kr) \end{pmatrix}, \qquad m < 0. \tag{4.7}$$

In order to apply the spectral boundary condition, one has previously to make the transformation [14]

$$\psi = e^{-i(\varphi/2)\gamma^5}\hat{\psi}, \tag{4.8}$$

obtaining

$$\hat{\psi}_m = \frac{1}{\sqrt{4\pi}}\begin{pmatrix} e^{i(m+1/2)\varphi}J_{m+\alpha}(kr) \\ -e^{i(m+1/2)\varphi}J_{m+\alpha+1}(kr) \end{pmatrix}, \qquad m \geqslant 0,$$

$$\hat{\psi}_m = \frac{1}{\sqrt{4\pi}}\begin{pmatrix} e^{i(m+1/2)\varphi}J_{-m-\alpha}(kr) \\ -e^{i(m+1/2)\varphi}J_{-m-\alpha-1}(kr) \end{pmatrix}, \qquad m < 0. \tag{4.9}$$

Now, these boundary conditions amount to

$$J_{m+\alpha+1}(kR) = 0, \qquad m+\tfrac{1}{2} > 0,$$

$$J_{-m-\alpha}(kR) = 0, \qquad m+\tfrac{1}{2} < 0; \tag{4.10}$$

that is

$$J_{m+\alpha}(kR)=0, \qquad J_{m-\alpha}(kR)=0, \qquad m=1,2,\dots. \tag{4.11}$$

We can therefore write (4.2) as

$$\begin{aligned}\langle\mu\rangle &= \exp\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\ln\frac{(k^-_{mn}-iM)(-k^-_{mn}-iM)(k^+_{mn}-iM)(-k^+_{mn}-iM)}{(k_{mn}-iM)(-k_{mn}-iM)(k_{mn}-iM)(-k_{mn}-iM)}\right\}\\ &= \exp\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\ln\frac{\left(k^{+2}_{mn}+M^2\right)\left(k^{-2}_{mn}+M^2\right)}{\left(k^2_{mn}+M^2\right)\left(k^2_{mn}+M^2\right)}\right\},\end{aligned} \tag{4.12}$$

where $k_{m,n}$ and $k^{\pm}_{m,n}$ are the same as appeared in the scalar case.

The computation now is almost identical to that we have already done in sect. 3 and we readily conclude that in the free spinor case

$$\langle\mu\rangle=\left(\frac{M^2}{\Lambda^2}\right)^{(\pi^2-2)\alpha^2/16}. \tag{4.13}$$

Again, the power dependence on Λ shows the renormalizability of the spinor disorder variable. As one should expect, $\langle\mu\rangle$ vanishes in the zero mass limit.

Let us now compute the order parameter, $\langle\sigma\rangle$, defined in sect. 2:

$$\langle\sigma\rangle=N\int[\mathrm{D}\psi][\mathrm{D}\bar\psi]\exp\left[-\int \mathrm{d}^2z\{i\bar\psi\,(\not\partial-i\alpha\gamma^5\not A)\psi-iM\bar\psi\psi\}\right], \tag{4.14}$$

where A_μ is given by (2.18) and $\alpha=\frac{1}{2}$, as explained above.

The functional integral is straightforward, giving

$$\langle\sigma\rangle=\mathrm{Det}\left[\frac{i(\not\partial-i\alpha\gamma^5\not A)-iM}{i\not\partial-iM}\right]. \tag{4.15}$$

We are, therefore, concerned with the eigenvalue equation (again inside a circular box of radius R)

$$\left[i(\not\partial-i\alpha\gamma^5\not A)-iM\right]\psi=\lambda\psi, \qquad \lambda=\lambda_0-iM, \tag{4.16}$$

which can be written as

$$\begin{pmatrix}-iM & -D_1+iD_2\\ \tilde D_1+i\tilde D_2 & -iM\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\lambda\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{4.17}$$

where $D_\mu = \partial_\mu - i\alpha A_\mu$ and $\tilde{D}_\mu = \partial_\mu + i\alpha A_\mu$. This equation is equivalent to

$$(-D_1 + iD_2)(\tilde{D}_1 + i\tilde{D}_2)\psi_1 = \lambda_0^2\psi_1,$$

$$\psi_2 = \frac{1}{\lambda_0}(\tilde{D}_1 + i\tilde{D}_2)\psi_1. \tag{4.18}$$

Again, for every ψ, there exists a $\psi' = \gamma^5\psi$, satisfying the equation

$$[-i(\partial\!\!\!/ - i\alpha\gamma^5 A\!\!\!/) - iM]\psi' = \lambda'\psi', \qquad \lambda' = -\lambda_0 - iM. \tag{4.19}$$

We can now write (4.18) in the "vortex gauge" as

$$\left[-\left(\partial_r^2 + \frac{1}{r}\partial_r\right) + \frac{1}{r^2}(m^2 - \alpha^2)\right]\psi_1 = \lambda_0^2\psi_1, \qquad m = 0, \pm 1, \dots . \tag{4.20}$$

For $m \neq 0$, requiring integrability at the origin, we obtain the solutions

$$\psi_m = \frac{1}{\sqrt{4\pi}}\begin{pmatrix} e^{im\varphi}J_{\sqrt{m^2-\alpha^2}}(kr) \\ \frac{1}{\lambda_0}(\tilde{D}_1 + i\tilde{D}_2)e^{im\varphi}J_{\sqrt{m^2-\alpha^2}}(kr)\end{pmatrix}, \qquad m \neq 0, \tag{4.21}$$

along with the corresponding ψ'_m.

Since we no longer have charge-conjugation invariance in (4.14), we can use the Dirichlet boundary condition

$$J_{\sqrt{m^2-\alpha^2}}(k'R) = 0, \qquad m = \pm 1, \pm 2, \dots, \tag{4.22}$$

obtaining the $m \neq 0$ contribution to $\langle\sigma\rangle$:

$$C = \exp\left\{2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\ln\frac{(-k'_{mn} - iM)(k'_{mn} - iM)}{(-k_{mn} - iM)(k_{mn} - iM)}\right\} = \exp\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\ln\frac{k'^2_{mn} + M^2}{k^2_{mn} + M^2}\right\}. \tag{4.23}$$

Using (3.7) again, we obtain

$$k'_{mn} = k_{mn} + \frac{\alpha^2}{2kR^2} - \frac{\pi\alpha^2}{4Rm} + O(\alpha^4). \tag{4.24}$$

When using (3.9), we have to consider in the present case that, since the angular momentum is $m = kR\sin\varphi$, the lower limit in the φ integral is $1/\Lambda R$.

Introducing (4.24) in (4.23), we find, using (3.9) in the $R \to \infty$ limit,

$$C \sim \exp\{-\ln R\}. \tag{4.25}$$

The proportionality constant is finite and the $O(\alpha^4)$ terms do not change the result.

For $m = 0$, we have an attractive potential in (4.20) which, apart from the bound states give the same finite contributions as before.

We conclude, therefore, that

$$\langle\sigma\rangle \sim \exp\{-\ln R\} = 0, \tag{4.26}$$

which is consistent with the fact that we are dealing with a free theory.

It is worth remarking that the use of spectral boundary conditions in this case would reproduce the result (4.26).

5. The construction of kink operators

In this section we are explicitly going to obtain the kink operators μ in the case of bosons and fermions.

An elegant formalism to implement this is the zeta function formalism, which has been used to calculate rigorously the determinant of the Schwinger model using the 't Hooft method [15]. Here we will follow another path for reconstructing the operators μ directly in Fock space.

Let us start with the free scalar case. Consider the euclidean Green functions

$$\begin{aligned}\langle\mu(0)\prod_{i=1}^{n}\phi(x_i)\phi^*(y_i)\rangle &= N\int[\mathrm{D}\phi][\mathrm{D}\phi^*]\exp\left[-\int \mathrm{d}^2t\left\{(D_\mu\phi)^*(D_\mu\phi)+M^2\phi^*\phi\right\}\right]\\ &\quad\times\prod_{i=1}^{n}\phi(x_i)\phi^*(y_i)\\ &=\langle\mu\rangle\left\langle\prod_{i=1}^{n}\phi(x_i)\phi^*(y_i)\right\rangle_{A_\mu}.\end{aligned} \tag{5.1}$$

The free field correlation functions in the external A_μ, are clearly sums over all contractions involving the basic Green functions $\langle\phi(x)\phi^*(y)\rangle_{A_\mu}$. In other words, the operator μ is, after suitable renormalization, of the form

$$\mu = \exp:\left\{\text{bilinear}(a, b, a^\dagger, b^\dagger)\right\}:, \tag{5.2}$$

where a and b are annihilation operator of particles and antiparticles. This bilinear exponential dependence, could also be expected from the formal expression (2.3).

The bilinear expression contains density terms $a^\dagger a$, $b^\dagger b$, as well as fluctuation terms ab, $a^\dagger b^\dagger$.

We first calculate the euclidean Green functions, directly using the $(R\to\infty)$ continuous eigenfunctions (3.5),

$$\begin{aligned}\langle\mu(0)\phi(x)\phi^*(y)\rangle &= G(x,y)_{A_\mu}\\ &=\frac{1}{2\pi}\int_0^\infty\frac{k\,\mathrm{d}k}{k^2+M^2}\sum_{m=0}^{\infty}\mathrm{e}^{i(m+\alpha)(\varphi_x-\varphi_y)}J_{m+\alpha}(kr_x)J_{m+\alpha}(kr_y)\\ &\quad+\frac{1}{2\pi}\int_0^\infty\frac{k\,\mathrm{d}k}{k^2+M^2}\sum_{m=1}^{\infty}\mathrm{e}^{i(-m+\alpha)(\varphi_x-\varphi_y)}J_{m-\alpha}(kr_x)J_{m-\alpha}(kr_y),\end{aligned}\tag{5.3}$$

where $0\leqslant\varphi\leqslant 2\pi$. In the above expression we have gauge transformed (3.5) back to the "string gauge", a transformation inverse to (3.3).

The euclidean Green function is periodic in α, with period 1 and for the split into two sums we have assumed $0\leqslant\alpha<1$.

Using the result [13]

$$\int_0^\infty\frac{k\,\mathrm{d}k}{k^2+M^2}J_\nu(kr_x)J_\nu(kr_y)=I_\nu(Mr^<)K_\nu(Mr^>),\tag{5.4}$$

where I_ν and K_ν are modified Bessel functions and $r^>$ ($r^<$) is the largest (smallest) of r_x and r_y, as well as the integral representations [13]

$$\begin{aligned}I_\nu(z)&=\frac{1}{2\pi i}\int_C\mathrm{e}^{z\cosh\omega-\nu\omega}\,\mathrm{d}\omega,\\ K_\nu(z)&=\tfrac12\int_{-\infty}^{\infty}\mathrm{e}^{-z\cosh v-\nu v}\,\mathrm{d}v,\end{aligned}\tag{5.5}$$

where C is a complex path beginning at $-i\pi+\infty$ and ending at $i\pi+\infty$, we obtain (for $r_x<r_y$, without loss of generality)

$$\begin{aligned}G(x,y)_{A_\mu}=\frac{1}{2i(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}v\oint_C\mathrm{d}\omega\Bigg[&\sum_{m=0}^{\infty}\mathrm{e}^{(m+\alpha)(i\varphi_x-i\varphi_y-v-\omega)}\\ &+\sum_{m=1}^{\infty}\mathrm{e}^{(-m+\alpha)(i\varphi_x-i\varphi_y+v+\omega)}\Bigg]\mathrm{e}^{Mr_x\cosh\omega}\mathrm{e}^{-Mr_y\cosh v}.\end{aligned}\tag{5.6}$$

Taking the lower limit of the v integral to be $-a$ and choosing the curve C, to lie on the right of Re $\omega = a$, one is in the region of convergence of the series. Therefore,

$$G(x, y)_{A_\mu} = \frac{1}{2i(2\pi)^2} \lim_{a\to\infty} \int_{-a}^{\infty} \mathrm{d}v \oint_{C_a} \mathrm{d}\omega \left[\frac{\mathrm{e}^{-\alpha(v+\omega-i\varphi_x+i\varphi_y)}}{1-\mathrm{e}^{(i\varphi_x-i\varphi_y-v-\omega)}} - \frac{\mathrm{e}^{\alpha(v+\omega+i\varphi_x-i\varphi_y)}}{1-\mathrm{e}^{(i\varphi_x-i\varphi_y+v+\omega)}} \right] \mathrm{e}^{Mr_x \cosh\omega} \mathrm{e}^{-Mr_y\cosh v} . \tag{5.7}$$

In the first integral, we now substitute $v \to -v$, and in the second, $\omega \to -\omega$, obtaining

$$\int_{-\infty}^{\infty} \mathrm{d}v \left[\oint_{C_a} \mathrm{d}\omega - \oint_{C'_a} \mathrm{d}\omega \right] \frac{\mathrm{e}^{\alpha(i\varphi_x - i\varphi_y + v - \omega)}}{1-\mathrm{e}^{(i\varphi_x - i\varphi_y + v - \omega)}} \mathrm{e}^{Mr_x\cosh\omega} \mathrm{e}^{-Mr_y\cosh v} , \tag{5.8}$$

where C'_a is the reflected contour.

Joining the paths, leads to an ω integral of the form

$$\int_{i\pi-\infty}^{i\pi+\infty} \mathrm{d}\omega - \int_{-i\pi-\infty}^{-i\pi+\infty} \mathrm{d}\omega + \oint_{C''} , \tag{5.9}$$

where C'' is a rectangle with sides $2a$ and $2\pi i$, centered in the origin. The residue at $\omega = v$ in the C'' integral gives the free Green function $G_0(x, y)$.

Therefore, the basic euclidean Green function is

$$G(x, y)_{A_\mu} = \frac{1}{2i(2\pi)^2} (\mathrm{e}^{-i\alpha\pi} - \mathrm{e}^{i\alpha\pi}) \int_{-\infty}^{\infty} \mathrm{d}v \times \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\mathrm{e}^{\alpha(i\varphi_x - i\varphi_y + v - \omega)}}{1+\mathrm{e}^{(i\varphi_x - i\varphi_y + v - \omega)}} \mathrm{e}^{-Mr_x\cosh\omega} \mathrm{e}^{-Mr_y\cosh v} + G_0(x, y) . \tag{5.10}$$

The Green function in the physical region is obtained by analytic continuation. For example, a contribution in the time-like region is obtained by substituting $r \to ir$, $i\varphi \to \chi$, where χ is the rapidity, $(x^0 = r\cosh\chi,\ x^1 = r\sinh\chi)$.

The result, after the change of variables $v \to -v + i\varphi_y$, $\omega \to -\omega + i\varphi_x$, is

$$G(x, y)_{A_\mu} = -\frac{\sin\alpha\pi}{(2\pi)^2} \int_{-\infty}^{\infty} \mathrm{d}v \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\mathrm{e}^{\alpha(\omega - v)}}{1+\mathrm{e}^{(\omega - v)}} \mathrm{e}^{-iMr_x\cosh(\omega-\chi_x)} \mathrm{e}^{-iMr_y\cosh(v-\chi_y)} + G_0(x, y) , \tag{5.11}$$

which is to be identified with

$$\langle 0|\mu(0)\phi(x)\phi^*(y)|0\rangle = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{\sqrt{p^0}}\int_{-\infty}^{\infty}\frac{\mathrm{d}q^1}{\sqrt{q^0}}\langle 0|\mu(0)|a^\dagger(p)b^\dagger(q)\rangle \mathrm{e}^{-ip\cdot x}\mathrm{e}^{-iq\cdot y} + G_0(x,y). \tag{5.12}$$

Hence, we obtain

$$\langle 0|\mu(0)|a^\dagger(\theta_p)b^\dagger(\theta_q)\rangle = -\frac{\sin\alpha\pi}{2\pi}\frac{\mathrm{e}^{\alpha(\theta_p-\theta_q)}\mathrm{e}^{-(\theta_p-\theta_q)/2}}{\cosh\left[\frac{1}{2}(\theta_p-\theta_q)\right]}, \tag{5.13}$$

where θ is the rapidity variable in momentum space, ($p_1 = M\cosh\theta$, $p_0 = M\sinh\theta$).

The other kernels in the time-like region are obtained by crossing transformation, $\theta \to \theta + i\pi$:

$$\langle b^\dagger(\theta_p)|\mu(0)|b^\dagger(\theta_q)\rangle = \frac{\sin\alpha\pi}{2\pi}\mathrm{e}^{i\alpha\pi}\frac{\mathrm{e}^{\alpha(\theta_p-\theta_q)}\mathrm{e}^{-(\theta_p-\theta_q)/2}}{\sinh\left(\frac{1}{2}\left[\theta_p-\theta_q\right]+i\varepsilon\right)}, \tag{5.14a}$$

$$\langle a^\dagger(\theta_q)|\mu(0)|a^\dagger(\theta_p)\rangle = \frac{\sin\alpha\pi}{2\pi}\mathrm{e}^{-i\alpha\pi}\frac{\mathrm{e}^{\alpha(\theta_p-\theta_q)}\mathrm{e}^{-(\theta_p-\theta_q)/2}}{\sinh\left(\frac{1}{2}\left[\theta_p-\theta_q\right]+i\varepsilon\right)}, \tag{5.14b}$$

$$\langle b^\dagger(\theta_p)a^\dagger(\theta_q)|\mu(0)|0\rangle = -\frac{\sin\alpha\pi}{2\pi}\frac{\mathrm{e}^{\alpha(\theta_p-\theta_q)}\mathrm{e}^{-(\theta_p-\theta_2)/2}}{\cosh\left(\frac{1}{2}\left[\theta_p-\theta_q\right]\right)}. \tag{5.14c}$$

The space-like region is obtained by shifting the respective χ's by $\frac{1}{2}i\pi$ and the results for the kernel are identical to (5.13), (5.14), as they should be.

The resulting expression for μ, therefore, is

$$\mu(x) = \mathrm{e}^{:K_\alpha(x):}, \tag{5.15a}$$

with

$$K_\alpha(0) = \frac{\sin\alpha\pi}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\theta_p\int_{-\infty}^{\infty}\mathrm{d}\theta_q \mathrm{e}^{\alpha(\theta_p-\theta_q)}\mathrm{e}^{-(\theta_p-\theta_q)/2} \times\left\{\frac{1}{\sinh\left(\frac{1}{2}\left[\theta_p-\theta_q\right]+i\varepsilon\right)}\left[\mathrm{e}^{-i\alpha\pi}a^\dagger(\theta_q)a(\theta_p)+\mathrm{e}^{i\alpha\pi}b^\dagger(\theta_p)b(\theta_q)\right] - \frac{1}{\cosh\left(\frac{1}{2}\left[\theta_p-\theta_q\right]\right)}\left[b(\theta_q)a(\theta_p)+a^\dagger(\theta_p)b^\dagger(\theta_q)\right]\right\}. \tag{5.15b}$$

It shows the basic property transformation under charge conjugation,

$$CK_\alpha C^\dagger = K_{1-\alpha}, \tag{5.16}$$

which was to be expected on the basis of the change of the string potential $A_\mu \to -A_\mu$, and the subsequent reduction of $-\alpha$ in (5.3) to the unit interval $-\alpha = 1 - \alpha (\mathrm{mod}\,1)$.

Let us now consider the fermion case. The eigenfunction representation analog to (5.3) is

$$G(x, y) = \sum_m \int_0^\infty k\,\mathrm{d}k \left[\frac{\psi_m(k, x)\psi_m^\dagger(k, y)}{k - iM} + \frac{\gamma^5\psi_m(k, x)\psi_m^\dagger(k, y)\gamma^5}{-k - iM} \right], \tag{5.17}$$

where the "string gauge" eigenfunctions analog to (4.7) are given by

$$\psi_m(k, x) = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} \mathrm{e}^{i(m+\alpha)\varphi} J_{m+\alpha}(kr) \\ -\mathrm{e}^{i(m+\alpha+1)\varphi} J_{m+\alpha+1}(kr) \end{pmatrix}, \qquad m \geqslant 0,$$

$$\psi_m(k, x) = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} \mathrm{e}^{i(m+\alpha)\varphi} J_{-m-\alpha}(kr) \\ -\mathrm{e}^{i(m+\alpha+1)\varphi} J_{-m-\alpha-1}(kr) \end{pmatrix}, \qquad m < 0. \tag{5.18}$$

Consider, for example, the (1.1) component, $G_{11} = iMG_{\mathrm{scalar}}$, where G_{scalar} is given by (5.3). The result in the time-like region follows from (5.13) and (5.14). It now, has to be compared with

$$\langle 0|\mu(0)\psi(x)\bar{\psi}(y)|0\rangle_{11} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}p^1}{\sqrt{p^0}} \int_{-\infty}^{\infty} \frac{\mathrm{d}q^1}{\sqrt{q^0}} u(p)\bar{v}(q)\mathrm{e}^{-ip\cdot x}\mathrm{e}^{-iq\cdot y}$$

$$\times \langle 0|\mu(0)|b^\dagger(q)a^\dagger(p)\rangle + G_0(x, y). \tag{5.19}$$

In our γ-representation, the spinors are

$$u = \sqrt{\tfrac{1}{2}M} \begin{pmatrix} \mathrm{e}^{\theta/2} \\ \mathrm{e}^{-\theta/2} \end{pmatrix}, \qquad v = \sqrt{\tfrac{1}{2}M} \begin{pmatrix} \mathrm{e}^{\theta/2} \\ -\mathrm{e}^{-\theta/2} \end{pmatrix},$$

with

$$\bar{v} = \sqrt{\tfrac{1}{2}M}\,(-\mathrm{e}^{-\theta/2}\mathrm{e}^{\theta/2}), \qquad \bar{u} = \sqrt{\tfrac{1}{2}M}\,(\mathrm{e}^{-\theta/2}\mathrm{e}^{\theta/2}). \tag{5.20}$$

We obtain, therefore, in the present case

$$\langle 0|\mu(0)|b^\dagger(\theta_q)a^\dagger(\theta_p)\rangle = i\frac{\sin\alpha\pi}{2\pi} \frac{\mathrm{e}^{\alpha(\theta_p - \theta_q)}}{\cosh\left(\frac{1}{2}[\theta_p - \theta_q]\right)}. \tag{5.21}$$

The discussion of the other components is consistent with this result. We obtain, in this case,

$$K_\alpha(0)=\frac{\sin\alpha\pi}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\theta_p\int_{-\infty}^{\infty}\mathrm{d}\theta_q e^{\alpha(\theta_p-\theta_q)}$$

$$\times\left\{\frac{1}{\sinh\left(\frac{1}{2}\left[\theta_p-\theta_q\right]+i\varepsilon\right)}\left[e^{-i\alpha\pi}a^\dagger(\theta_q)a(\theta_p)-e^{i\alpha\pi}b^\dagger(\theta_p)b(\theta_q)\right]\right.$$

$$\left.+\frac{i}{\cosh\left(\frac{1}{2}\left[\theta_p-\theta_q\right]\right)}\left[b(\theta_q)a(\theta_p)+a^\dagger(\theta_p)b^\dagger(\theta_q)\right]\right\}. \tag{5.22}$$

The charge conjugation for $\langle\mu(0)\psi(x)\bar{\psi}(y)\rangle$, is again equivalent to $\alpha\to 1-\alpha$.

The present form of the operator is not convenient if one wants to calculate the short-distance behaviour of the μ correlation function. In that case, one uses an (justifiable) infinite resummation [16], to obtain

$$\mu(x)=\frac{\mathrm{N}[e^{-i2\alpha\sqrt{\pi}\varphi(x)}]}{\langle\mathrm{N}[e^{-i2\alpha\sqrt{\pi}\varphi(x)}]\rangle}, \tag{5.23}$$

where

$$\varphi(0)=\frac{i}{4\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}\theta_p\int_{-\infty}^{\infty}\mathrm{d}\theta_q\left[\frac{1}{\cosh\left(\frac{1}{2}\left[\theta_p-\theta_q\right]\right)}\left(a(\theta_p)b(\theta_q)-b^\dagger(\theta_q)a^\dagger(\theta_p)\right)\right.$$

$$\left.+\frac{1}{\sinh\left(\frac{1}{2}\left[\theta_p-\theta_q\right]+i\varepsilon\right)}\left(a^\dagger(\theta_p)a(\theta_q)-b^\dagger(\theta_p)b(\theta_q)\right)\right]. \tag{5.24}$$

Here, N denotes the normal ordering which subtracts only the two-point vacuum expectation value, i.e., $\mathrm{N}[\phi^2]=\phi^2-\langle\phi^2\rangle$.

This normal ordering is finite in the unit interval $0\leqslant\alpha<1$ for which our formalism has been derived. The reader easily recognizes in φ the charge-conjugation antisymmetric sine-Gordon potential belonging to the free massive Dirac field. We could also have obtained this resummation in the framework of euclidean functional integration by using the "euclidean bosonization" of our lagrangian [12].

The short-distance singularities of the μ correlation functions are now exposed. They are identical to those of an exponential massless free field. The dimension of μ is simply α^2.

In the scalar case the discussion of the short-distance behaviour and the zero-mass limit is more difficult because the corresponding potential,

$$j^\mu=\frac{1}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_\nu\varphi,$$

has a logarithm square behavior and thus does not approach a zero-mass Bose field.

Our formalism of relating string gauge fields to derivative of discontinuities in sections of vector bundles only allows one to deal with strings of strength α smaller than one. Higher strings have to be obtained by a limiting procedure of amalgamating several strings in different positions at the end of our calculation. In the Minkowski-space operator formalism such a procedure corresponds to defining a disorder operator with strength $\beta = n\alpha$ by "fusing" n μ's of the "reduced" strength α via a short-distance limiting procedure

$$\mu_{n\alpha}(x) = \text{leading term in } \lim_{x_i \to x} \prod_{i=1}^{n} \mu_\alpha(x_i). \tag{5.25}$$

In the spinor case one expects this extended formalism to yield a dimension $(n\alpha)^2$ and not $n \cdot \alpha^2$, i.e., the leading c-numbers in the short-distance expansion will involve positive powers.

The order-operator σ in the T_c^+ phase can be directly constructed within the Lieb-Mattis-Schulz approach to the Lenz-Ising model. In the scaling limit it takes the form [16–18]

$$\sigma = :\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\mathrm{d}p^1}{p^0} \left(\mathrm{e}^{-ip\cdot x} a(p) + \mathrm{e}^{ip\cdot x} a^\dagger(p)\right) \mathrm{e}^{K(a, a^\dagger)}:, \tag{5.26}$$

where K is the same exponential operator as appearing in μ. Using the field-theoretical, short-distance expansion of Kadanoff and Wilson [19] one may also obtain σ as the leading operator in the product of a Majorana spinor with μ [11]. Within the present context of the euclidean functional approach, the operator σ emerges from the study of certain correlation functions of γ^5 gauge fields with $\alpha = \frac{1}{2}$. We will not give this derivation of (5.26) in this paper.

6. Lenz-Ising field theory and non-trivial Z_2 bundles

The formalism of Bohm-Aharonov is only applicable in the case of complex matter fields. For self-conjugate fields, e.g. real scalar fields or Majorana spinor fields, there is no minimal coupling to a vector potential. However an antiperiodic boundary condition for "sections" still remains meaningful and leads to the disorder operator of the form (5.15) with $\alpha = \frac{1}{2}$ and $a = b$. Physically more interesting are the operators μ and σ in the case of a Majorana field. They are again obtained from (5.22) for $a = b$ and $\alpha = \frac{1}{2}$. Starting from the Lieb-Mattis-Schulz [20] formulation of the Ising model, it is fairly easy [16, 17] to see that they are the disorder and order variables in the scaling limit $T \to T_c^+$. For the scaling limit from below T_c, the order and disorder variables interchange roles apart from a subtle point concerning the appearance of Z_2 "spurion" operators by which one has to enlarge the Fock space of the Majorana fields. Let us now have a close look at the mathematical framework for such a Z_2 gauge theory. The non-triviality of the topology for real (Majorana) fields forces us to use the language of fibre bundle theory. For example in the calculation of the $\mu(0)$ one point function we start from the pointed space $R^2 - \{0\}$ which can be covered by two neighbourhoods (φ = polar angle):

$$U_1: \qquad -\varepsilon < \varphi < \pi + \varepsilon,$$

$$U_2: \qquad \pi - \varepsilon < \varphi < \varepsilon. \tag{6.1}$$

A flat non-trivial real vector bundle is obtained by taking as patching functions

$$\begin{aligned} g(\varphi) &= -1 \qquad \text{in } (\pi-\varepsilon, \pi+\varepsilon) \in U_1 \cap U_2, \\ g(\varphi) &= 1 \qquad \text{in } (-\varepsilon, \varepsilon) \in U_1 \cap U_2 \end{aligned} \tag{6.2}$$

which take values in the gauge group Z_2. In terms of real sections this leads to antiperiodic boundary conditions. Differently from the complex case, there exists no non-vanishing (at each φ) standard section with the help of which one could trivialize this vector bundle and therefore no minimal coupling to A_μ. Thus the functional integration approach to the Lenz-Ising field theory illustrates a point which had been particularly emphasized by Wu and Yang [7]. These authors used an example of a monopole situation for which, in principle, the Dirac language of singular strings was applicable. This is not the case in our illustration: the language of non-trivial vector bundles becomes a necessity. In addition, the Lenz-Ising model was not invented to illustrate this point and hence it is more "natural".

Note that the case of kinks in Z_N models for $N \geqslant 3$ requires a complex vector bundle and hence can be formulated in the standard language of Bohm-Aharonov potentials A_μ. Note, furthermore that on the basis of the Z_N selection rules in the broken symmetry phase

$$\langle \mu(x_1) \cdots \mu(x_n) \rangle \Rightarrow \begin{cases} = 0, & n \neq 0, \quad \text{mod } N, \\ \neq 0, & n = 0, \quad \text{mod } N \end{cases}$$

(the last property being related to the appearance of the identity operator in the leading short-distance term for the product of N operators μ), one expects such a model for $N \geqslant 3$ to be dynamically always non-trivial: an antiparticle has to be a bound state of $N-1$ particles [21],

$$\mu^\dagger \sim \mu^{N-1}.$$

Finally, we would like to comment on the role of "doubling" in the Ising model. It is clear from our formalism that the formation of a Dirac field from two Majorana ones allows, on the one hand, the use of a vector potential and hence the bosonization formalism (5.23) and, on the other hand, leads to a square relationship between the μ-correlation functions of the simple and the doubled model. The latter fact is read off from the determinantal representation of correlation functions. This doubling, in fact, was used as a construction of the Ising correlation functions using the sine-Gordon potential [16]. In the past this construction appeared somewhat artificial compared with the more direct kink construction via Clifford algebra techniques [17]. However, now, with the appearance of a unified functional integral formalism, it is a logical and simplifying step, because the formalism of exponentials of sine-Gordon potentials is much more developed and certain properties like the short-distance singularities, the massless limit and the very elusive Kadanoff-Ceva selection rules on one base line are natural consequences of the doubled model.

The formalism of doubling also extends to the Lenz-Ising σ operator. There, using the language of Fock-space operator, one needs to double the ψ factor in front of μ as well; our σ in the Dirac (doubled) model is not the same object as the square (in the sense of the functional integral representation) of σ in the Majorana model. Without going into details, we quote the answer for these two hermitian operators which had been obtained previously without using functional integration [11]:

$$\mu_D(x) = \tfrac{1}{2}(:e^{i\sqrt{\pi}\varphi}: + :e^{-i\sqrt{\pi}\varphi}:) = :\cos\sqrt{\pi}\,\varphi(x):,$$

$$\sigma_D(x) = \frac{1}{2i}(:e^{i\sqrt{\pi}\varphi}: - :e^{-i\sqrt{\pi}\varphi}:) = :\sin\sqrt{\pi}\,\varphi(x):. \tag{6.3}$$

Their special form of sums of exponentials is related to the peculiarity of the case $\alpha = \frac{1}{2}$.

In the doubled version one loses the sign ambiguity, which was a property of the mixed functions in the simple model, and the operator become relatively local, instead of relatively dual.

7. Generalizations to non-abelian kinks

In the previous sections we have seen that the euclidean functional integration for two-dimensional kinks is the theory of matter fields in Bohm-Aharonov fluxes or in a more mathematical language: the theory of flat vector bundles over pointed R^2 which may or may not be topologically non-trivial. There exists, however, another approach to this problem which has been at times advocated by 't Hooft. The idea is to consider the singular gauge situation of Bohm-Aharonov as a limit of smooth gauge configurations. This point of view becomes useful in situations in which one is able to calculate generic determinants [22]. This is possible in two-dimensional massless spinor models with abelian or non-abelian gauge fields. Consider, e.g. a $U(n)$ massless spinor field in a non-abelian gauge field $\underline{A}_\mu$. Assume that the asymptotic behaviour of $\underline{A}_\mu$ is such that the "partial indices" (explained below)

$$n_i = 0,$$

i.e. an asymptotic behaviour of the form (φ = polar angle)

$$A_\mu \to iU^{-1}\partial_\mu U, \tag{7.1}$$

with

$$U(\varphi) \sim U^{(0)}(\varphi) = \begin{pmatrix} e^{if_1(\varphi)} & & 0 \\ & \ddots & \\ 0 & & e^{if_n(\varphi)} \end{pmatrix}, \tag{7.2}$$

with vanishing partial winding numbers

$$n_i = \int_0^{2\pi} f_i(\varphi)\,d\varphi = 0. \tag{7.3}$$

The equivalence relation used in (7.2) means that there exists a non-singular unitary transformation $V(\varphi)$ with

$$U(\varphi) = V^\dagger(\varphi) U^{(0)}(\varphi) V(\varphi). \tag{7.4}$$

The determinant on these configurations $\underline{A}_\mu$ has the form [22]

$$\ln \frac{\operatorname{Det} i\not{D}}{\operatorname{Det} i\not{\partial}} = -\Gamma = \int j_\mu^a A_\mu^a d^2x,$$

with

$$j_\mu^a = -\frac{1}{4\pi}\int d^2y\, \varepsilon^{\mu\nu} K_\nu^{ab}(x,y)\varepsilon^{\lambda\rho}F_{\lambda\rho}^b(y),$$

$$D_\mu^{ab} K_\mu^{bc}(x,y) = -\delta(x-y)\delta^{ac},$$

$$\varepsilon^{\mu\nu} D_\mu^{ab} K_\nu^{bc}(x,y) = 0. \tag{7.5}$$

Hence K is the spinor Green function in the adjoint representation ($K_\pm = K_1 \pm iK_2$)

$$\begin{pmatrix} 0 & -D_1 + iD_2 \\ D_1 + iD_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & K_- \\ -K_+ & 0 \end{pmatrix} = -\mathbb{1}\cdot\delta(x-y). \tag{7.6}$$

Consider as an example the disorder operator μ of an abelian (U(1)) massless Dirac field. The two-point function $\langle \mu_\alpha(x)\mu_\beta^\dagger(y)\rangle$ is obtained as the functional determinant evaluated in the vortex gauge:

$$A_\mu(z) = \varepsilon^{\mu\nu}\left[\alpha\frac{(z-x)_\nu}{(z-x)^2} - \beta\frac{(z-y)_\nu}{(z-y)^2}\right]. \tag{7.7}$$

An examination of the boundary condition shows that the determinant $e^{-\Gamma}$ remains finite in the limit $R\to\infty$ only if $\alpha=\beta$, in which case one obtains the result

$$\langle \mu_\alpha(x)\mu_\alpha^\dagger(y)\rangle = C e^{-\alpha^2 \ln m(x-y)^2}, \tag{7.8}$$

where m is regularization parameter.

The σ_α two-point function yields the same result from the functional determinant corresponding to the γ^5 (axial) vortex potential analogous to (7.7). Note that an euclidean axial potential or a mixed vector-axial vector potential leads to an $A_\pm$ with

$$A_+ \neq (A_-)^*.$$

Writing the Schwinger determinant

$$\Gamma = \frac{1}{2\pi}\int A_\mu(z)A_\mu(z)\,d^2z \tag{7.9}$$

in the $\pm$ representation [the general determinant (7.5) simplifies to this case in the abelian situation] one obtains the general mixed correlation function of σ_{α_i}'s and μ_{β_i}'s including the selection rule

$$\sum_i \alpha_i = 0, \qquad \sum_i \beta_i = 0. \tag{7.10}$$

In the non-abelian situation the μ's and σ's correspond to "non-abelian rotation strings" (γ^5 in the case of σ's) $\underline{A}_\mu = iU^{-1}\partial_\mu U$:

$$\mu = \mu(x;U), \qquad \sigma = \sigma(x;W). \tag{7.11}$$

The selection rules may be most easily expressed in terms of the logarithms (generators) of the U's and W's which we will call H and K:

$$\sum_i H_i = 0, \qquad \sum_i K_i = 0. \tag{7.12}$$

In addition to the non-abelian "half-space" phase factors occurring in the duality relations between μ's and σ's there is a new phenomenon which may be observed already on the level of commutation relations between μ's only:

$$\mu(x;U_1)\mu(y;U_2)=\begin{cases}\mu(y_i;U_2)\mu(x;U_2^{-1}U_1U_2), & x<y,\\ \mu(y;U_1U_2U_1^{-1})\mu(x;U_1), & x>y.\end{cases}\tag{7.13}$$

The results of this euclidean functional approach can now be related to the work of Sato et al. [8] on the relation of the Riemann-Hilbert problem with disorder operator in massless Dirac field theories with one essential difference however. The selection rules (7.12) lead to the vanishing of many ψ-$\bar{\psi}$ 3-point functions,

$$\left\langle\psi(x)\bar{\psi}(y)\prod_{i=1}^{n}\mu(x_i;U_i)\right\rangle=\left\langle\prod_{i=1}^{n}\mu(x_i;U_i)\right\rangle\langle\psi(x)\bar{\psi}(y)\rangle_{\underline{A}_\mu}=0,\tag{7.14}$$

unless the selection rules (7.12) hold. Hence in most interesting cases the 3-point functions corresponding to the Riemann-Hilbert problem are related to the vacuum expectation values of μ's by infrared-divergent factors and hence do not constitute objects belonging to QFT.

8. Concluding remarks

In this work we show that the ideas of Kadanoff and 't Hooft on order-disorder duality, hitherto mainly used for lattice theories, have a natural extension to continuous field theories. Applied to the problem of kinks in two space-time dimensions they yield euclidean functional integrals for matter fields coupled to Bohm-Aharonov gauge potentials. In continuous QFT the rather subtle renormalization properties of kinks become inexorably linked with gauge invariance, the latter being responsible for the path independence which in turn yields the Lorentz covariant transformation properties of kinks. Another advantage of the continuous approach is that properties related to non-trivial topology become more clearly recognizable.

This work constitutes a first attempt to understand kinks (topological solitons) and disorder variables in continuous field theory outside the quasiclassical approach and hence many problems remain open. Among the important theoretical problems not discussed in this work is the structural relation between the euclidean functional integrals and the physical correlation functions, i.e. the Osterwalder-Schrader properties for mixed order-disorder correlation functions. We also left the perturbative construction of kinks in the broken symmetry phase and the generalizations to higher dimensional kinks for future investigations.

We are deeply saddened by the premature death of our friend and colleague Jorge André Swieca and we miss very much his clarifying participation in the writing of this paper.

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